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The Effect of an External Field on an Interface, Entropic Repulsion

Joel L. Lebowitz[1] and Christian Maes[1,2]

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We consider the effects of an external potential $-h\sum f(S_x)$ with $h>0$, f increasing, on the equilibrium state of a system with a Hamiltonian of the form

$$H^0(S)=\sum_{\langle xy\rangle}\Phi(S_x-S_y),\ S_x\in R,\ x\in Z^d,\ d\geqslant 3$$

Φ even and convex, e.g., $\Phi(t)=1/2t^2$ and $f(t)=\text{sign } t$. This can be thought of as a model of the interactions between a random interface $\{S_x\}$ and a "soft" wall. We show that the random surface is (entropically) repelled to infinity for all $h>0$, i.e., with probability one, $S_x\geqslant K$, for any $K\in R$.

KEY WORDS: Random interfaces; soft wall; entropic repulsion.

1. INTRODUCTION

We consider the simple cubic lattice Z^d in d dimensions, to each of whose sites $x\in Z^d$ we assign a spin variable with values $S_x\in R$. Let Λ, $V,...\subset Z^d$ denote finite cubes with center at the origin of the lattice. A configuration $S_V=\{S_x;x\in V\}$ in V is an element of $R^{|V|}$, where $|V|$ is the number of sites in V.

The energy for configurations in V with a given specified boundary condition $S_x=a$ if $x\notin V$ is assumed to have the form

$$H^0(S_V;a)=\sum_{\langle xy\rangle}\Phi(S_x-S_y)$$

[1] Department of Mathematics and Physics, Rutgers University, New Brunswick, New Jersey 08903.

[2] Aspirant NFWO, Instituut voor Theoretische Fysika, K.U. Leuven, B-3030 Leuven, Belgium.

The sum is over all pairs of nearest neighbors x, y at least one of which is in the region V. The function Φ is even and satisfies the condition

$$\int \exp[-\alpha\Phi(t)]\, dt < \infty, \qquad \forall\alpha > 0$$

The joint distribution of the spins in the region V is given by the normalized Gibbs measure

$$d\mu^0_{V,a} = [Z_{V,a}]^{-1} \exp[-H^0(S_V; a)] \prod_V dS_x$$

$Z_{V,a}$ is the partition function and dS_x is the usual Lebesgue measure on R. The inverse temperature is set equal to one. We will write

$$\langle \cdot \rangle^0_{V,a} = \int \cdot \, d\mu^0_{V,a}$$

for the expectation with respect to this measure.

The thermodynamic limit of the free energy

$$f_V = \frac{1}{|V|} \log Z_{V,a} \to f$$

always exists[1] and is independent of a. We will assume that an infinite-volume equilibrium state can be defined. This is always true for $d \geqslant 3$ when $\Phi(t) = \alpha t^2 + v(t)$, $\alpha > 0$, v convex; see Section 3. The limiting measure will of course depend (in a trivial way) on a. For the case $\Phi(t) = \frac{1}{2}t^2$, it is the massless harmonic crystal: the Gaussian field $\{S_x; x \in Z^d\}$ specified by its covariance $-\Delta^{-1}$ (Δ is the lattice Laplacian) and its mean a. Here $\langle \cdot \rangle^0_a$ will denote expectation with respect to the infinite-volume measure, i.e.,

$$\langle \cdot \rangle^0_a = \lim_V \langle \cdot \rangle^0_{V,a}$$

We now introduce Ising-like variables

$$\begin{aligned} \sigma_x = \operatorname{sign} S_x &= 1 \quad \text{if} \quad S_x \geqslant 0 \\ &= -1 \quad \text{if} \quad S_x \leqslant 0 \end{aligned}$$

and consider the probability distribution of the Ising variables $\{\sigma_x; x \in Z^d\}$ induced by the Gibbs field $\{S_x; x \in Z^d\}$ described above. In this way we obtain a degraded image of the original random field, i.e., the configurations $\{\sigma_x; x \in V\}$ reveal only whether the corresponding spins S_x are above or below the zero level. We call $\tilde{\mu}_a$ the induced measure.

One of the motivations to study the induced system is that situations of this type may arise naturally in some image reconstruction problems.[2] We note that the state $\tilde{\mu}_a$ will not (as we shall see) be Gibbsian with any finite-range interaction. This gives our induced system some of the characteristics of a nonequilibrium state, whose study is of great interest.[3] An additional feature of this induced system inherited from the original Gibbs system is the strong (critical) correlations among the $\{\sigma_x\}$. For the harmonic case an easy computation gives

$$\langle \sigma_0; \sigma_x \rangle_a^0 \approx 1/|x|^{d-2}, \qquad d \geq 3$$

so this is an explicitly treatable model of Ising spins with nonintegrable power law decay of the truncated pair correlation.

In this paper we will not go into a more detailed analysis of this harmonic Ising model (HIM as we like to call it), but rather investigate the effects on $\tilde{\mu}_a$ of a perturbation by an external magnetic field. Let

$$H^{h,\Lambda}(S_V; a) = H^0(S_V; a) - h \sum_\Lambda \sigma_x, \qquad h > 0, \quad \Lambda \subset V$$

and let

$$d\mu_{V,a}^{h,\Lambda} = [Z_{V,a}(h, \Lambda)]^{-1} \exp[-H^{h,\Lambda}(S_V; a)] \prod_V dS_x$$

be the perturbed measure with expectations

$$\langle \cdot \rangle_{V,a}^{h,\Lambda} = \int \cdot \, d\mu_{V,a}^{h,\Lambda} = \left[\left\langle \exp\left(h \sum_\Lambda \sigma_x \right) \right\rangle_{V,a}^0 \right]^{-1} \left\langle \cdot \exp\left(h \sum_\Lambda \sigma_x \right) \right\rangle_{V,a}^0$$

The superscript h, Λ is a reminder that a magnetic field has been applied only to the spins $\{\sigma_x; x \in \Lambda\}$. If $V = \Lambda$, we will simply write $\langle \cdot \rangle_{V,a}^h$.

This model has some relevance in the statistical mechanics of random surfaces. The variables $\{S_x\}$ may be viewed as describing the height of an interface above a zero level. Adding the perturbation is then like introducing an interaction of the interface with an external potential. We shall call, with a certain abuse of language, any potential of the form

$$-h \sum f(S_x)$$

with f any (nonconstant) nondecreasing function, a (soft) wall. The characteristics of the wall are described by h and the function f. Our main result is then that the interface is repelled to infinity by the wall independent of the form of the wall! It would be essentially flat ($S_x = a$, $d > 2$) if the wall were absent ($h=0$). Our result may be considered as a generalization of the work by Bricmont *et al.*[4] These authors considered the case where the external potential corresponds to a rigid wall

$$f(t)=0 \qquad \text{if} \quad t \geqslant 0$$

$$= -\infty \qquad \text{if} \quad t<0$$

This is the $h \uparrow \infty$ limit of the case $f(t) = \text{sign}\ t - 1$ or the $\alpha \uparrow \infty$ limit of the case $f(t) = -\exp(-\alpha t)$ (see also remark 3.2). The surface is constrained in this case to fluctuate above the rigid wall and Bricmont *et al.*[4] showed that it escapes to infinity.

The underlying mechanism in our case is the same as in Ref. 4. The surface wants to be "above the wall" (energy requirement) and wants to stay "away from the wall" in order to have more freedom to fluctuate (entropic repulsion); cf. discussion in Ref. 4.

Our arguments center around the invariance of the Hamiltonian H^0, under a noncompact continuous symmetry group. The existence of an infinite-volume Gibbs state in this case can be expected only for $d \geqslant 3$.[1,5] We show that even in this case any perturbation of the kind described above pushes the system to infinity.

In the proofs frequent use will be made of the FKG inequalities. In order that the measures defined above satisfy the FKG inequality, it is sufficient that Φ is convex and smooth enough.[6,7]

In Section 2 we present our results and proofs. Section 3 is a discussion of these results. The nature of the induced state on the fuzzy variables $\{\sigma_x\}$, $\sigma_x = \text{sign}\ S_x$, is further discussed. Examples are given and the assumptions of Section 2 are reviewed. A remark is made about the relevance of our results to the Liouville model in field theory. The case $S_x \in Z$ (as in the discrete harmonic crystal or the SOS model) is also discussed.

2. RESULTS AND PROOFS

We assume for simplicity (see, however, Remark 3.3) that Φ is C^2 and of the form

$$\Phi(t) = \alpha t^2 + v(t), \qquad \alpha > 0 \quad \text{and } v \text{ convex}$$

The next lemma gives the difference in free energy between the perturbed and the unperturbed system for the case $f(S_x) = \text{sign}\ S_x = \sigma_x$.

Lemma 1:

(a) $$\lim_V \frac{1}{|V|} \log \left\langle \exp h \sum_V \sigma_x \right\rangle^0_{V,a} = h$$

(b) $$\lim_\Lambda \frac{1}{|\Lambda|} \log \left\langle \exp h \sum_\Lambda \sigma_x \right\rangle^0_{a} = h$$

Proof:

$$\exp(h\,|V|) \geqslant \left\langle \exp h \sum_V \sigma_x \right\rangle^0_{V,a} \geqslant [\exp(h\,|V|)] \left\langle \prod_V \chi(S_x \geqslant 0) \right\rangle^0_{V,a} \tag{1}$$

where $\chi(\cdot)$ is the characteristic function of the event $\cdot$.

We also have for any $k \geqslant 0$

$$\left\langle \prod_V \chi(S_x \geqslant 0) \right\rangle^0_{V,a} \geqslant \left\langle \prod_V \chi(2k \geqslant S_x \geqslant 0) \right\rangle^0_{V,a} = \left\langle \prod_V \chi(|S_x| \leqslant k) \right\rangle^0_{V,a-k} \tag{2}$$

The last equality reflects the presence of a zero mode, i.e., the interaction is invariant (up to a change in the boundary condition) under a uniform shift of all the spin values.

For part (a) of the lemma it is sufficient to show that

$$\lim_V \frac{1}{|V|} \log \left\langle \prod_V \chi(|S_x| \leqslant k) \right\rangle^0_{V,a-k} = \lim_V \frac{1}{|V|} \log \left\langle \prod_V \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \tag{3}$$

and

$$\sup_k \lim_V \frac{1}{|V|} \log \left\langle \prod_V \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} = 0 \tag{4}$$

To prove (3) we write

$$\begin{aligned} &\left| \log \frac{\langle \prod \chi(|S_x| \leqslant k) \rangle^0_{V,a-k}}{\langle \prod \chi(|S_x| \leqslant k) \rangle^0_{V,0}} \right| \\ &= \left| \int_{a-k}^0 d\xi \frac{d}{d\xi} \log \langle \prod \chi(|S_x| \leqslant k) \rangle^0_{V,\xi} \right| \\ &= \left| \sum_{\partial_i V} \int_{a-k}^0 d\xi \langle \Phi'(S_x - \xi) \rangle^0_{V,\xi}(k_V) \right| \\ &\leqslant \sum_{\partial_i V} \left| \int_{a-k}^0 d\xi \langle \Phi'(S_x - \xi) \rangle^0_{V,\xi}\,(k_V) \right| \end{aligned} \tag{5}$$

where $\langle \cdot \rangle^0_{V,\xi}(k_\Lambda)$ refers to the unperturbed measure with cutoff $-k \leqslant S_x \leqslant k$, for all $x \in \Lambda \subset V$, and $\partial_i V$ are the sites in V forming the inner boundary. Since $\Phi(t)$ is convex and even, $\Phi'(t)$ is increasing and odd. Therefore, the rhs of (5) can be bounded for $k > a$ by

$$|\partial V| \int_{a-k}^{0} d\xi\, \Phi'(k-\xi) = |\partial V|\, [\Phi(2k-a) - \Phi(k)]$$

It follows from this bound that (3) is satisfied.

To prove (4), we observe that for any $V' \subset V$ such that $y \in V'$ and $V' \backslash y = V''$, we have

$$\begin{aligned}
&\left\langle \prod_{V'} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \\
&\quad = [1 - \langle \chi(|S_y| > k)\rangle^0_{V,0}(k_{V''})] \left\langle \prod_{V''} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \\
&\quad \geqslant [1 - k^{-2} \langle S_y^2 \rangle^0_{V,0}(k_{V''})] \left\langle \prod_{V''} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0}
\end{aligned}$$

where the equality follows from the general definition and the inequality is a Chebyshev inequality.

Using Brascamp–Lieb inequalities,[1,8] it is possible to bound $\langle S_y^2 \rangle^0_{V,0}(k_{V''})$ by $\langle S_y^2 \rangle^{\text{harm}}_{V,0}$, where harm refers to the case $\Phi(t) = \alpha t^2$. For $d \geqslant 3$ the latter is uniformly bounded in the volume, so that we have that $\langle S_y^2 \rangle^0_{V,0}(k_{V''}) \leqslant c$. Therefore,

$$\left\langle \prod_{V'} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \geqslant (1 - c/k^2) \left\langle \prod_{V''} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \tag{6}$$

Starting from $V' = V$, we can iterate (6) untill $V'' = \varnothing$. At the end we get

$$\left\langle \prod_{V} \chi(|S_x| \leqslant k) \right\rangle^0_{V,0} \geqslant (1 - c/k^2)^{|V|}$$

Now (4) is easily obtained.

This completes the proof of part (a).

To prove (b) it suffices to show that

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \log \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle^0_a = 0 \tag{7}$$

This will be reduced to part (a) of this lemma by using FKG to establish the following lower bound: for all $b \in R$

$$\left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_a^0 \geqslant \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_{\Lambda,b}^0 [\langle \chi(S_0 \geqslant 0) \rangle_{a-b}^0]^{|\partial\Lambda|} \tag{8}$$

To prove (8), we introduce a constraint for the spins at the outer boundary of Λ:

$$\left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_a^0 \geqslant \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \prod_{\partial\Lambda} \chi(S_x \geqslant b) \right\rangle_a^0$$

As a consequence of the DLR characterization[9] of a Gibbs state, we have that the

$$\text{rhs} = \left\langle \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_{\Lambda, \{S_x; x \in \partial\Lambda\}}^0 \prod_{\partial\Lambda} \chi(S_x \geqslant b) \right\rangle_a^0$$

where

$$\left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_{\Lambda, \{S_x; x \in \partial\Lambda\}}^0$$

is the usual finite-volume expectation with respect to $d\mu_{\Lambda, \{S_x; x \in \partial\Lambda\}}^0$ of a function of the spins in Λ with boundary conditions specified by the set $\{S_x; x \in \partial\Lambda\}$.

By the convexity of Φ and the FKG property,

$$\left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_{\Lambda, \{S_x; x \in \partial\Lambda\}}^0$$

is an increasing function of the S_x, $x \in \partial\Lambda$. Hence,

$$\left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_a^0 \geqslant \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_{\Lambda,b}^0 \left\langle \prod_{\partial\Lambda} \chi(S_x \geqslant b) \right\rangle_a^0$$

The desired lower bound is now obtained by again using the FKG inequality for the second factor in the rhs. This ends the proof of the lemma. ▮

Lemma 2:

(a) $\lim_{V} \langle \sigma_0 \rangle_{V,a}^h = 1$

(b) $\lim_{\Lambda} \langle \sigma_0 \rangle_a^{h,\Lambda} = 1$

Proof. By convexity in h, we can differentiate[10] both sides of Lemma 1 to obtain, for any $h>0$,

$$(\mathrm{a}')\quad \lim_V \frac{1}{|V|}\sum_V \langle\sigma_x\rangle_{V,a}^h = 1$$

$$(\mathrm{b}')\quad \lim_\Lambda \frac{1}{|\Lambda|}\sum_\Lambda \langle\sigma_x\rangle_a^{h,\Lambda} = 1$$

By FKG, $\langle\sigma_x\rangle_a^{h,\Lambda}$ is an increasing function in $h_y = h$ if $y\in\Lambda$, $h_y=0$ if $y\notin\Lambda$. Hence, $\langle\sigma_x\rangle_a^{h,\Lambda}$ is an increasing function in Λ and $\lim_\Lambda \langle\sigma_x\rangle_a^{h,\Lambda}$ exists and is independent of $x\in Z^d$,

$$1 \geqslant \lim_{\Lambda'} \langle\sigma_0\rangle_a^{h,\Lambda'} \geqslant \langle\sigma_x\rangle_a^{h,\Lambda} \tag{9}$$

for $x\in\Lambda$. We now sum (9) over $x\in\Lambda$ and divide by $|\Lambda|$. Taking the limit $\Lambda\uparrow Z^d$, we get, using (b'), that the rhs is equal to 1, which proves (b). Part (a) follows from (b) since by FKG for $\Lambda\subset V$

$$1 \geqslant \langle\sigma_0\rangle_{V,a}^h \geqslant \langle\sigma_0\rangle_{V,a}^{h,\Lambda}$$

We can therefore first take the limit $V\uparrow Z^d$ and then $\Lambda\uparrow Z^d$. This completes the proof of Lemma 2. ∎

It follows from Lemma 2 that $S_0 \geqslant K$ with probability one, for any $K\leqslant 0$. It also holds for $K>0$. Indeed, if there was a nonzero probability that $S_x<K$ for $x\in\partial\Lambda$ one could condition on this event and using the DLR equation arrive at a contradiction to Lemma 2. We thus have:

Theorem 1.

(a) $\lim_V \langle\chi(S_0<K)\rangle_{V,a}^h = 0$

(b) $\lim_\Lambda \langle\chi(S_0<K)\rangle_a^{h,\Lambda} = 0 \qquad$ for any $K\in R$, $\quad h>0$

3. ADDITIONAL REMARKS

3.1. The results of Section 2 also apply to the general case where

$$\sigma_x = f(S_x), \qquad x\in Z^d$$

with f any (nonconstant) nondecreasing function.

Indeed, consider an increasing function with

$$\begin{aligned} f_b(x) &= r && \text{if} \quad x\geqslant b \\ &= -r && \text{if} \quad x<-b, \quad b>0 \end{aligned}$$

Section 2 can then be copied in its entirety (with some zeros replaced by $\pm b$, and some ones replaced by r). The general case then follows by FKG.

3.2. Recently there has been some interest in the case

$$\Phi(t)=\tfrac{1}{2}t^2, \qquad f(t)=-\exp(-\alpha t), \qquad \alpha>0$$

The corresponding system in field theory is known as the Liouville model. There have been various efforts to quantize the Liouville theory (see e.g., Refs. 11 and 12 and references therein) and to give a well-defined perturbation theory for it. Our results show that in the context of Gibbs states for a lattice system with bounded pure boundary conditions, a nontrivial Liouville theory does not exist, i.e., the expected value of S_0 is $+\infty$. This, however, does not exclude the possibility that a non-translation-invariant state can be defined as a limit of finite-volume Gibbs states with boundary conditions $a_V \downarrow -\infty$ which are volume dependent.

3.3. We have taken Φ to be convex. This, when Φ is smooth enough, e.g., C^2, is sufficient to have the FKG inequalities for the different measures of Section 1. The Brascamp–Lieb inequalities[8] imply that the model obtained by choosing $\Phi(t)=\alpha t^2+v(t)$, $\alpha>0$ and $v(t)$ convex, is dominated by the harmonic crystal, which is well defined in the infinite-volume limit for $d\geqslant 3$. This form for Φ then satisfies all our assumptions. However, the proofs of Section 2 can be repeated also for other forms of Φ as long as (4) holds. More details on the statistical mechanics of anharmonic lattices can be found in Ref. 1.

3.4. In Lemma 1 of Section 2 we have shown that

$$\lim_{\Lambda} \frac{1}{|\Lambda|} \log \left\langle \prod_{\Lambda} \chi(S_x \geqslant 0) \right\rangle_a^0 = 0$$

This, together with the application of the FKG inequalities, has been the main ingredient in the proofs. We would like to strengthen (7) to obtain a sharper large-deviation behavior for our models.[10]

Consider the massless harmonic crystal $\{S_x; x\in Z^d\}$, $\langle S_0\rangle=0$. Since any sum of jointly Gaussian random variables is again Gaussian, it is easy to compute that, for large V and $a>0$,

$$\left\langle \chi\left(\frac{1}{|V|}\sum_V S_x \geqslant a\right)\right\rangle_0^0 \approx \exp(-c_a\,|V|/\chi_V) \tag{10}$$

where the susceptibility

$$\chi_V=\sum_V \langle S_0; S_x\rangle \approx |V|^{2/d} \qquad \text{as} \quad V\uparrow Z^d$$

The expression (10) gives an upper bound to $\langle \prod_V \chi(S_x \geqslant a)\rangle_0^0$. We conjecture that there exists a lower bound of the same form, i.e.,

$$\left\langle \prod_V \chi(S_x \geqslant a)\right\rangle_0^0 \approx \exp(-c_a L^{d-2}), \qquad L^d = |V|$$

3.5. If we add a mass term $-m^2 \sum_V S_x^2$ to the Hamiltonian H^0, the spins will obviously remain bounded in the thermodynamic limit, even when the external potential $-h \sum_V f(S_x)$ has been added. We could then study the (h, m) phase diagram.

For example, in the harmonic case with $f = \text{sign}$ we get qualitatively the picture shown in Fig. 1. At the critical point $(0, 0)$ an uncountable number of pure phases appear, each corresponding to different ways of approaching this point. See Ref. 13 for a general discussion of this phenomenon.

3.6. Consider again the measure $\tilde{\mu}_a$ defined in Section 1. We have shown that the presence of a magnetic field h forces the spins $\{\sigma_x\}$ to be $+1$ with probability one, for any $h > 0$. This implies that $\tilde{\mu}_a$ cannot be a Gibbs measure with any decent Hamiltonian, e.g., one in which the interaction of a single spin with all the other spins is uniformly bounded.

3.7. In one and two dimensions the unperturbed system typically shows too large a fluctuations to define an equilibrium state: $\langle |S_0| \rangle \uparrow +\infty$. By using the FKG inequalities it is easy to derive that

$$\langle S_0 \rangle_{V,0}^{h,A} \geqslant e^{-h} \sinh h \, \langle |S_0| \rangle_{V,0}^0$$

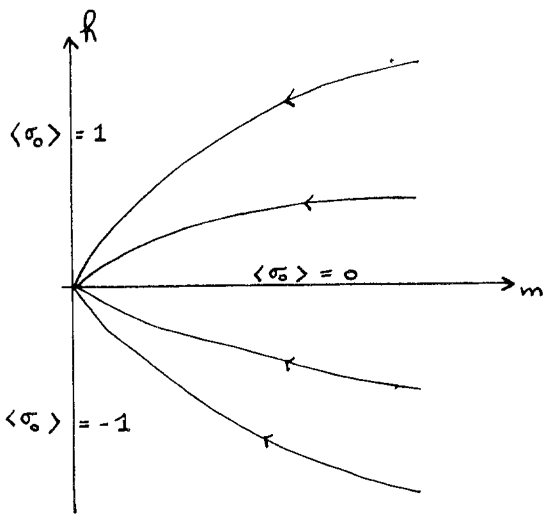

Fig. 1.

In one dimension the transfer operator method can be used to calculate the rate of escape of S_x as some restoring force goes to zero. For example, for the massive harmonic crystal in one dimension, one finds that the (even) moments go to infinity as the mass goes to zero at the same rate for the unperturbed and the perturbed measure. The interesting case is therefore in $d=3$, where $\langle |S_0| \rangle_0^0$ is bounded, but $\langle S_0 \rangle_0^{h,\Lambda} \to \infty$.

3.8. So far we have been solely concerned with the case $S_x \in R$. While more abstract settings of the problem are possible, let us just consider one other case: $S_x \in Z$. The methods of Section 2 can be easily applied to the discrete harmonic crystal and the SOS model. The SOS model is defined by taking $\Phi(t) = |t|$. Various estimates (especially Theorem 3.2) of Ref. 4 can be used to simplify the proof of our Lemma 1. In Ref. 4, information was obtained about the rate at which the average height of the random surface goes to infinity if the constraint $S_x \geqslant 0$ is imposed on these systems.

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